UNSTEADY FLOW OF A MAXWELL FLUID IN A POROUS RECTANGULAR DUCT

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ABSTRACT. The analytic solution for the unsteady magnetohydrodynamic (MHD) flow of Maxwell fluid in long porous rectangular cross-section is studied. Two flow problems are considered: (i) Flow in an oscillating rectangular duct and (ii) Flow in a duct induced by an oscillating pressure gradient. The problems are solved by applying the double finite Fourier sine and Laplace transforms by taking into account the modified Darcy's law. The effects of magnetic parameter and porosity of medium on the velocity profile, the corresponding tangential tensions and volume flow rate for both the problems, and the influence of various material parameters on the velocity profile for the second problem are presented graphically and discussed.

Key Words: *Maxwell fluid, MHD flow, Oscillating rectangular duct, Oscillating pressure gradient, Volume flow rate*

1. INTRODUCTION

The research of non-Newtonian fluids plays an important role in many engineering and industrial applications due to their behavior. In order to describe the behavior of non-Newtonian fluids, numerous constitutive models have been proposed. A rate type model, which is widely used is due to Maxwell. W. Akhtar et al. [1] discussed the unsteady flow of a Maxwell fluid induced by a constantly accelerating plate between two side walls perpendicular to the plate and obtained exact solutions for the velocity field and tangential stresses by means of the Fourier sine transforms. T. Hayat et al. [2]studied the unsteady flow of a Maxwell fluid caused by a suddenly moved plane wall between two side walls perpendicular to the plane and obtained closed form solution employing the Fourier sine transforms. M.E. Erdogan [3] discussed the unsteady flow of a viscous fluid due to the cosine and sine oscillations of a plane wall and L. Zheng et al. [4] developed exact solutions for generalized Maxwell fluid flow due to oscillatory and constantly accelerating plate. C.K. Chen et al. [5] and W. Akhtar and M. Nazar [6] obtained exact solutions for the flows in a circular duct for Maxwell fluids and generalized Maxwell fluids respectively. H.T. Qi et al. [7] obtained solutions corresponding to the unsteady flows of fractional Maxwell fluid in a duct of rectangular cross-section analytically and M. Nazar et al. [8] extended the problem for flow through oscillating rectangular duct.

The study of flow through porous media is of fundamental importance in geo mechanics, biomechanics and industry. Flows through porous media include flow of water through rocks, regulation of skin and filtration of fluids. Porous passages with rectangular cross sections are useful for cooling of engineering systems. Following Henry, mathematical descriptions of liquid flow in porous media are based on Darcy's law. A.K. Johri et al. [9] discussed oscillating flow of a viscous liquid in a porous rectangular cross-section under the influence of periodic pressure gradient. They derived expressions for velocity distribution, volume flow rate and drag in the duct. T.G. Prasuna et al. [10] examined unsteady flow of a visco elastic fluid through a porous media between two impermeable parallel plates employing Laplace transformation technique and computed expressions for the flow rate and shear stress on the walls.

The presence of an external magnetic field that effects the motion of non-Newtonian fluids, for example blood, is very important. G. Ramamurty and B. Shankar [11] discussed the effect of Magnetohydrodynamic on blood flow through a porous channel. B. Muck [12] investigated the magnetohydrodynamic liquid metal flow around a square cylinder placed in a rectangular duct. S. Smolentsev [13] discussed magnetohydrodynamic flows in a conducting rectangular duct witha non-conducting flow channel insert in a constant transverse magnetic field. T. Hayat et al. [14] obtained the analytic solution for unsteady magnetohydrodynamic flow in a rotating non-Newtonian fluid through a porous medium. M. Khan et al. [15]presented MHD transient flows in a channel of rectangular cross-section with porous medium. R.M. Mohyuddin [16] discussed the Newtonian problem of an unsteady MHD flow past an infinite oscillating porous plate with general free stream velocity using Laplace transform technique.

The purpose of this paper is to present analytic solutions for the MHD flow of Maxwell fluid through porous oscillating rectangular duct in the absence of pressure gradient and in a rectangular duct of oscillating pressure gradient. The expressions for the velocity, corresponding tangential tensions and volume flow rate are determined by means of double finite Fourier sine and Laplace transforms. To solve the problem we have used the usual condition, the first time derivative of the velocity is zero at time $t = 0$.

2. Governing Equations

The Cauchy stress tensor, τ, for an incompressible Maxwell fluid is given by the constitutive equation

$$\tau = -pI + S \quad and \quad S + \lambda \dot{S} = 2\mu D, \qquad (1)$$

where p is the hydrostatic pressure, μ is the dynamic viscosity, I is the identity tensor, S is the extra stress tensor, λ is the relaxation time, $\dot{S}$ is the material time derivative of Sand D is the deformation tensor.

Consider an incompressible Maxwell fluid at rest in a duct of rectangular cross-section, whose sides are at x=0, x=d, y=0and y=h. At time t=0+, the duct begins to oscillate along z-axis.

The velocity field is of the form [8] $\mathbf{V}=\mathbf{V}(x,y,t)=w(x,y,t)\hat{\mathbf{k}},$ (2)

where $\hat{\mathbf{k}}$ denotes the unit vector along the z-coordinate direction. We will assume that the extra stress S depends only on x, y and t, that is

$$S=S(x,y,t). \tag{3}$$

A uniform Magnetic field $J\times B$ is applied to the fluid, where $J=\sigma(\mathbf{E}+\mathbf{V}\times\mathbf{B})$ is the current density, σ is the electric conductivity of the fluid, $\mathbf{E}$ is the electric field, $\mathbf{V}$ is the velocity field, $\mathbf{B}$ is the total magnetic field, so that $\mathbf{B}=\mathbf{B}_o+\mathbf{b}$, where $\mathbf{B}_o$ is the intensity of applied magnetic field and $\mathbf{b}$ is the induced magnetic field. In the present analysis, the external electric field and the induced magnetic field are assumed to be negligible such that the magnetic Reynolds number is small. It is also assumed that the magnetic field is perpendicular to the velocity field.

Thus the Lorentz force due to magnetic field becomes

$$\mathbf{J}\times\mathbf{B}=-\sigma B_0^2\mathbf{V}. \tag{4}$$

The Darcy's resistance in a Maxwell fluid satisfies the following expression

$$(1+\lambda\frac{\partial}{\partial t})R=-\frac{\mu\phi}{k}\mathbf{V}, \tag{5}$$

Where ϕ is the porosity of the medium, μ is the dynamic viscosity, K is the permeability of the porous medium and R is the Darcy's resistance.

The unsteady motion of MHD fluid is governed by the following equations

$$\nabla\bullet\mathbf{V}=0, \tag{6}$$

$$(1+\lambda\frac{\partial}{\partial t})\tau=\mu(\nabla\cdot\mathbf{V}), \tag{7}$$

$$\rho\frac{\partial\mathbf{V}}{\partial t}=-\nabla P+\nabla\bullet\tau+\mathbf{J}\times\mathbf{B}+R, \tag{8}$$

where ρ is the density of fluid.

3. Flow in an oscillating rectangular duct

Substituting Eqs. (1)- (5) into Eq. (8) and taking into account the initial condition $\mathbf{S}(x,y,0)=\mathbf{0}$, we get $S_{xx}=S_{xy}=S_{yy}=0$, and the governing equation

$$(1+\lambda\frac{\partial}{\partial t})\frac{\partial w(x,y,t)}{\partial t}=\nu[\frac{\partial^2 w(x,y,t)}{\partial x^2}+\frac{\partial^2 w(x,y,t)}{\partial y^2}] -\frac{\sigma\beta_0^2}{\rho}(1+\lambda\frac{\partial}{\partial t})w(x,y,t)-\frac{\nu\,\phi}{k}w(x,y,t), \tag{9}$$

where $\nu=\dfrac{\mu}{\rho}$ is the kinematic viscosity.

We consider the following initial and boundary conditions

$$w(x,y,0)=\frac{\partial w(x,y,0)}{\partial t}=0 \text{ for } (x,y)\in(0,d)\times(0,h), \tag{10a}$$

$$w(0,y,t)=w(d,y,t)=w(x,0,t)=w(x,h,t)=U\cos(\omega t) \text{ for all } t, \tag{10b}$$

or

$$w(x,y,0)=\frac{\partial w(x,y,0)}{\partial t}=0 \text{ for } (x,y)\in(0,d)\times(0,h), \tag{11a}$$

$$w(0,y,t)=w(d,y,t)=w(x,0,t)=w(x,h,t)=U\sin(\omega t) \text{ for all } t. \tag{11b}$$

We denote by $u(x,y,t)$ the solution of problem (9), (10a), (10b) and by $v(x,y,t)$ the solution of problem (9), (11a), (11b).

By introducing the function

$$\mathrm{F}(\mathrm{x},\mathrm{y},\mathrm{t})=\mathrm{u}(\mathrm{x},\mathrm{y},\mathrm{t})+\mathrm{i}\ \mathrm{v}(\mathrm{x},\mathrm{y},\mathrm{t}), \tag{12}$$

we obtain the following problem

$$(1+\lambda\frac{\partial}{\partial t})\frac{\partial F(x,y,t)}{\partial t}=\nu\,[\frac{\partial^2 F(x,y,t)}{\partial x^2}+\frac{\partial^2 F(x,y,t)}{\partial y^2}] -H(1+\lambda\frac{\partial}{\partial t})F(x,y,t)-\frac{\nu\,\phi}{k}F(x,y,t), \tag{13}$$

$$F(x,y,0)=\frac{\partial F(x,y,0)}{\partial t}=0 \text{ for } (x,y)\in(0,d)\times(0,h), \tag{14}$$

$$F(0,y,t)=F(d,y,t)=F(x,0,t)=F(x,h,t)=Ue^{i\omega t} \text{ for all } t, \tag{15}$$

where $H=\dfrac{\sigma\beta_0^2}{\rho}$ is the magnetic parameter.

4. Calculation of the velocity field

The solution of the problem (13)-(15) will be obtained by means of the double finite Fourier sine and Laplace transforms.

We denote by

$$F_{mn}(t)=\int_0^d\int_0^h F(x,y,t)\sin(\frac{m\pi}{d}x)\sin(\frac{n\pi}{h}y)dxdy, \quad \text{for } m,n=1,2,3,\cdots$$

the double finite Fourier sine transform of function $F(x,y,t)$.

Let us take

$$I=\int_0^d\int_0^h(\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2})\sin(\lambda_m x)\sin(\mu_n y)dxdy$$

$$=-(\lambda_m^2+\mu_n^2)F_{mn}(t)+\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\ \mu_n}\times(\lambda_m^2+\mu_n^2)\,U e^{i\,\omega\,t}, \tag{16}$$

where $\lambda_m=\dfrac{m\pi}{d}$ and $\mu_n=\dfrac{n\pi}{h}$.

Applying the double finite Fourier sine transform to Eq. (13) and using boundary conditions (15), we obtain

$$\lambda \frac{d^2F_{mn}(t)}{dt^2}+(1+\lambda H)\frac{dF_{mn}(t)}{dt}+(\nu\lambda_{mn}+H+\frac{\nu\phi}{k})F_{mn}(t)$$
$$=\nu\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}\lambda_{mn}Ue^{i\omega t},\ t>0 \quad (17)$$

where

$$\lambda_{mn}=\lambda_m^2+\mu_n^2, \qquad m,n=1,2,3,\cdots \quad (18)$$

The double finite Fourier sine transforms $F_{mn}(t)$ of $F(t)$ has to satisfy the initial conditions

$$F_{mn}(0)=\frac{dF_{mn}(0)}{dt}=0 \text{ for } (x,y)\in(0,d)\times(0,h). \quad (19)$$

Applying the Laplace transform to Eq. (17) and using Eq. (19), we obtain

$$\bar{F}_{mn}(s)=\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}\frac{\nu\lambda_{mn}U}{s-i\omega}$$
$$\times\frac{1}{\lambda s^2+(1+\lambda H)s+\nu\lambda_{mn}+H+\frac{\nu\phi}{k}}, \quad (20)$$

where $\bar{F}_{mn}(s)=\int_0^\infty F_{mn}(t)e^{-st}dt$ is the Laplace transform of the function $F_{mn}(t)$ and s is the Laplace transform variable. We can write Eq. (20) in the following form

$$\bar{F}_{mn}(s)=\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}\frac{U}{s-i\omega}$$
$$-\frac{U[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}$$
$$\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{mn})}. \quad (21)$$

where

$$Z_{mn}=\nu\lambda_{mn}+H+\frac{\nu\varphi}{k}.$$

Let us take

$$\bar{L}_{mn}(s)=\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{mn})}$$
$$=\frac{A_{mn}}{s-i\omega}+(1-A_{mn})$$
$$\times\{\frac{(s+\frac{1+\lambda H}{2\lambda})}{(s+\frac{1+\lambda H}{2\lambda})^2-(\frac{b_{mn}}{2\lambda})^2}+\frac{1+\lambda H+2\lambda\omega i}{b_{mn}}$$
$$\times\frac{\frac{b_{mn}}{2\lambda}}{(s+\frac{1+\lambda H}{2\lambda})^2-(\frac{b_{mn}}{2\lambda})^2}\}, \quad (22)$$

where

$$\mathrm{A}_{mn}=\mathrm{c}_{mn}+\mathrm{i}\ \mathrm{d}_{mn}, \quad (23)$$

c_{mn}, d_{mn} and b_{mn} are defined by

$$c_{mn}=\frac{\lambda\omega^2\left(\lambda\omega^2-\nu\lambda_{mn}-2(H+\frac{\nu\phi}{k})\right)}{\left(-\lambda\omega^2+\nu\lambda_{mn}+H+\frac{\nu\phi}{k}\right)^2+\omega^2(1+\lambda H)^2}$$
$$+\frac{(H+\frac{\nu\phi}{k})Z_{mn}+\omega^2(1+\lambda H)^2}{\left(-\lambda\omega^2+Z_{mn}\right)^2+\omega^2(1+\lambda H)^2} \quad (24)$$

$$d_{mn}=\frac{\nu\omega\lambda_{mn}(1+\lambda H)}{\left(-\lambda\omega^2+Z_{mn}\right)^2+\omega^2(1+\lambda H)^2}, \quad (25)$$

and

$$b_{mn}=\sqrt{(1+\lambda H)^2-4\lambda Z_{mn}}. \quad (26)$$

Inverse Laplace transform of function $\bar{L}_{mn}(s)$ given in Eq. (22) is

$$L_{mn}(t)=A_{mn}\exp(i\omega t)+(1-A_{mn})\exp\left(-\frac{(1+\lambda H)t}{2\lambda}\right)$$
$$\times\left\{\cosh(\frac{b_{mn}}{2\lambda}t)+\frac{1+\lambda H+2\lambda\omega i}{b_{mn}}\sinh(\frac{b_{mn}}{2\lambda}t)\right\}. \quad (27)$$

By applying the inverse Laplace transform and then inverse Fourier sine transform to Eq. (21) and using Eq. (27), we obtain

$$F(x,y,t)=\frac{4U}{dh}\exp(i\omega t)\sum_{m,n=1}^{\infty}\sin(\lambda_m x)\sin(\mu_n y)$$
$$-\frac{4U}{dh}\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}\sin(\lambda_m x)\sin(\mu_n y)$$
$$\times\{A_{mn}\exp(i\omega t)+(1-A_{mn})\times\exp\left(-\frac{(1+\lambda H)t}{2\lambda}\right)$$
$$\times\left\{\cosh\left(\frac{b_{mn}}{2\lambda}t\right)+\frac{1+\lambda H+2\lambda\omega i}{b_{mn}}\sinh\left(\frac{b_{mn}}{2\lambda}t\right)\right\}\}, \quad (28)$$

or

$$F(x,y,t)=U\exp(i\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\{A_{pq}\exp(i\omega t)+(1-A_{pq})\times\exp\left(\frac{-(1+\lambda H)t}{2\lambda}\right)$$
$$\times\left(\cosh(\frac{b_{pq}}{2\lambda}t)+\frac{1+\lambda H+2\lambda\omega i}{b_{pq}}\sinh(\frac{b_{pq}}{2\lambda}t)\right)\}, \quad (29)$$

where $\lambda_p=(2m+1)\frac{\pi}{d},\ \mu_q=(2n+1)\frac{\pi}{h},\ p=2m+1$ and $q=2n+1$.

By taking $t\to\infty$ *and* $1+\lambda H>0$ in Eq. (29), we obtain the following steady-state solution

$$F_s(x,y,t)=U\exp(i\omega t)-\frac{16U\exp(i\omega t)}{dh}\times\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}A_{pq}. \qquad (30)$$

The velocity fields corresponding to both cosine and sine oscillations of the duct obtained from Eq. (29) are

$$u(x,y,t)=U\cos(\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}\times\{c_{pq}\cos(\omega t)-d_{pq}\sin(\omega t)+\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\times\Big((1-c_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)+\frac{(1-c_{pq})(1+\lambda H)+2\lambda\omega d_{pq}}{b_{pq}}\times\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\}, \qquad (31)$$

and

$$v(x,y,t)=U\sin(\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}\times\left(\begin{array}{l}c_{pq}\sin(\omega t)+d_{pq}\cos(\omega t)\\ -\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\Big(d_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)\\ +\frac{d_{pq}(1+\lambda H)-2\lambda\omega(1-c_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\end{array}\right). \qquad (32)$$

From Eqs. (31) and (32), the steady-state solutions for both cosine and sine oscillations are

$$u_s(x,y,t)=U\cos(\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)\sin(\mu_q y)}{\lambda_p\mu_q}c_{pq}\cos(\omega t)+\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}d_{pq}\sin(\omega t), \qquad (33)$$

$$v_s(x,y,t)=U\sin(\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}c_{pq}\sin(\omega t)-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}d_{pq}\times\cos(\omega t), \qquad (34)$$

while

$$u_t(x,y,t)=-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}\times\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\Big\{(1-c_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)+\frac{(1-c_{pq})(1+\lambda H)+2\lambda\omega d_{pq}}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}, \qquad (35)$$

$$v_t(x,y,t)=-\frac{16U_0}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}\times\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\Big\{-d_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)-\frac{d_{pq}(1+\lambda H)-2\lambda\omega(1-c_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}, \qquad (36)$$

are the corresponding transient components for $1+\lambda H>0$.

5. Calculation for τ, the tangential tension

In our problem, we have $S_{xx}=S_{xy}=S_{yy}=0$, and

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau_1(x,y,t)=\mu\frac{\partial\omega(x,y,t)}{\partial x}, \qquad (37)$$

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau_2(x,y,t)=\mu\frac{\partial\omega(x,y,t)}{\partial y}, \qquad (38)$$

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\sigma(x,y,t)=2\lambda\Big(\tau_1\frac{\partial\omega}{\partial x}+\tau_2\frac{\partial\omega}{\partial y}\Big), \qquad (39)$$

where $\tau_1=S_{xz}$, $\tau_2=S_{yz}$ *and* $\sigma=S_{zz}$.

We denote by $\tau_{1c}(x,y,t)$, $\tau_{2c}(x,y,t)$ the tangential stresses for the cosine oscillations of the duct and $\tau_{1s}(x,y,t)$, $\tau_{2s}(x,y,t)$ the tangential stresses for the sine oscillations of the duct respectively.

If we introduce

$$\tau_1(x,y,t)=\tau_{1c}(x,y,t)+i\tau_{1s}(x,y,t), \qquad (40)$$

$$\tau_2(x,y,t)=\tau_{2c}(x,y,t)+i\tau_{2s}(x,y,t), \qquad (41)$$

into the above Eqs., we obtain

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau_1(x,y,t)=\mu\frac{\partial F(x,y,t)}{\partial x}, \qquad (42)$$

$$\Big(1+\lambda\frac{\partial}{\partial t}\Big)\tau_2(x,y,t)=\mu\frac{\partial F(x,y,t)}{\partial y}. \qquad (43)$$

By applying the Laplace transform to Eqs. (42) and (43), we obtain

$$\bar{\tau}_1(x,y,s)=\frac{\mu}{1+\lambda s}\frac{\partial\bar{F}(x,y,s)}{\partial x}, \qquad (44)$$

$$\bar{\tau}_2(x,y,s)=\frac{\mu}{1+\lambda s}\frac{\partial \bar{F}(x,y,s)}{\partial y}. \qquad (45)$$

From Eq. (21), we obtain

$$\bar{F}(x,y,s)=\frac{U}{s-i\omega}-\frac{4U}{dh}\sum_{m,n=1}^{\infty}\frac{[1-(-1)^m][1-(-1)^n]}{\lambda_m\mu_n}\times\frac{\sin(\lambda_m x)\sin(\mu_n y)}{\lambda_m\mu_n}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{pq})}, \qquad (46)$$

or

$$\bar{F}(x,y,s)=\frac{U}{s-i\omega}-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)\sin(\mu_q y)}{\lambda_p\mu_q}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{pq})}, \qquad (47)$$

where $p=2m+1,\ q=2n+1$.

Differentiating Eq. (47) w.r.t. x and y respectively, we get

$$\frac{\partial \bar{F}}{\partial x}(x,y,s)=-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{pq})}, \qquad (48)$$

$$\frac{\partial \bar{F}}{\partial y}(x,y,s)=-\frac{16U}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(\lambda s^2+(1+\lambda H)s+Z_{pq})}. \qquad (49)$$

Using Eq. (48) in Eq. (44) and Eq. (49) in Eq. (45), we have

$$\bar{\tau}_1(x,y,s)=-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(1+\lambda s)(\lambda s^2+(1+\lambda H)s+Z_{pq})}, \qquad (50)$$

$$\bar{\tau}_2(x,y,s)=-\frac{16\mu U_0}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}\times\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(1+\lambda s)(\lambda s^2+(1+\lambda H)s+Z_{pq})}. \qquad (51)$$

Let us take

$$\bar{G}_{pq}(s)=\frac{\lambda s^2+(1+\lambda H)s+H+\frac{\nu\phi}{k}}{(s-i\omega)(1+\lambda s)(\lambda s^2+(1+\lambda H)s+Z_{pq})}=\frac{B_{pq}}{s-i\omega}+\frac{C_{pq}}{1+\lambda s}-\frac{(\lambda e_{pq}+g_{pq})+i(\lambda f_{pq}+h_{pq})}{\lambda}\times\frac{s+\frac{1+\lambda H}{2\lambda}}{\left(s+\frac{1+\lambda H}{2\lambda}\right)^2-\left(\frac{b_{pq}}{2\lambda}\right)^2}+$$

$$\times\left\{\frac{2\lambda Z_{pq}(f_{pq}-\omega g_{pq})}{\lambda\omega b_{pq}}+\frac{\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}\right\}\frac{\frac{b_{pq}}{2\lambda}}{\left(s+\frac{1+\lambda H}{2\lambda}\right)^2-\left(\frac{b_{pq}}{2\lambda}\right)^2}$$

$$+i\left\{\frac{H+\frac{\nu\phi}{k}-2\lambda Z_{pq}(e_{pq}+\omega h_{pq})}{\lambda\omega b_{pq}}+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\right\}\frac{\frac{b_{pq}}{2\lambda}}{\left(s+\frac{1+\lambda H}{2\lambda}\right)^2-\left(\frac{b_{pq}}{2\lambda}\right)^2}, \qquad (52)$$

where

$$B_{pq}=e_{pq}+i\,f_{pq},\ C_{pq}=g_{pq}+i\,h_{pq}, \qquad (53)$$

e_{pq}, f_{pq}, g_{pq} and h_{pq} are given by

$$e_{pq}=\frac{\omega^2\left(\lambda^2(\omega^2+H^2+\nu\lambda_{pq}H)+1-\frac{2\lambda\nu\phi}{k}\right)}{[Z_{pq}-\lambda\omega^2(2+\lambda H)]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}+\frac{(H+\frac{\nu\phi}{k})Z_{pq}}{[Z_{pq}-\lambda\omega^2(2+\lambda H)]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}, \qquad (54)$$

$$f_{pq}=\frac{\omega[\lambda^2\omega^2\left(\frac{2\nu\phi}{k}+\nu\lambda_{pq}-\lambda(\omega^2+H^2)\right)]}{[Z_{pq}-\lambda\omega^2(2+\lambda H)]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}+$$

$$\frac{\omega[-\lambda\left(\omega^2+\frac{\nu^2\phi}{k}\lambda_{pq}+(H+\frac{\nu\phi}{k})^2\right)+\nu\lambda_{pq}]}{[Z_{pq}-\lambda\omega^2(2+\lambda H)]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}, \quad (55)$$

$$g_{pq}=\frac{-\phi\lambda}{k(\frac{\phi}{k}+\lambda_{pq})(1+\lambda^2\omega^2)},$$
$$h_{pq}=\frac{\phi\omega\lambda^2}{k(\frac{\phi}{k}+\lambda_{pq})(1+\lambda^2\omega^2)}. \quad (56)$$

Applying inverse Laplace transform to Eq. (52), we obtain

$$G_{pq}=B_{pq}\exp(i\omega t)+\frac{C_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$-\exp\left(-\frac{1+\lambda H}{2\lambda}t\right)$$
$$\times\left\{\left(\frac{(\lambda e_{pq}+g_{pq})+i(\lambda f_{pq}+h_{pq})}{\lambda}\right)\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right.$$
$$+\{\frac{2\lambda Z_{pq}(f_{pq}-\omega g_{pq})}{\lambda\omega b_{pq}}$$
$$+\frac{\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}\}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)$$
$$+i\{\frac{H+\frac{\nu\phi}{k}-2\lambda Z_{pq}(e_{pq}+\omega h_{pq})}{\lambda\omega b_{pq}}$$
$$\left.+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right)\}. \quad (57)$$

Applying the inverse Laplace transform to Eqs. (50) and (51) and using Eq. (57), we obtain

$$\tau_1(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\left(B_{pq}\exp(i\omega t)+\frac{C_{pq}}{\lambda}\exp(-\frac{t}{\lambda})-\exp\left(-\frac{1+\lambda H}{2\lambda}t\right)\right.$$
$$\times\left\{\frac{(\lambda e_{pq}+g_{pq})+i(\lambda f_{pq}+h_{pq})}{\lambda}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right.$$
$$+\{\frac{2\lambda Z_{pq}(f_{pq}-\omega g_{pq})+\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}$$
$$+i\{\frac{H+\frac{\nu\phi}{k}-2\lambda\left((\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(e_{pq}+\omega h_{pq})\right)}{\lambda\omega b_{pq}}$$
$$\left.\left.+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\}\right\}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\}\right), \quad (58)$$

$$\tau_2(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\left(B_{pq}\exp(i\omega t)+\frac{C_{pq}}{\lambda}\exp(-\frac{t}{\lambda})-\exp\left(-\frac{1+\lambda H}{2\lambda}t\right)\right.$$
$$\times\left\{\frac{(\lambda e_{pq}+g_{pq})+i(\lambda f_{pq}+h_{pq})}{\lambda}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right.$$
$$+\{\frac{2\lambda Z_{pq}(f_{pq}-\omega g_{pq})+\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}$$
$$+i\{\frac{H+\frac{\nu\phi}{k}-2\lambda\left((\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(e_{pq}+\omega h_{pq})\right)}{\lambda\omega b_{pq}}$$
$$\left.\left.+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\}\right\}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\}\right). \quad (59)$$

From Eqs. (58) and (59), we obtain the tangential tensions

$$\tau_{1c}(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\left(e_{pq}\cos(\omega t)-f_{pq}\sin(\omega t)+\frac{g_{pq}}{\lambda}\exp(-\frac{t}{\lambda})\right.$$
$$+\exp\left(-\frac{1+\lambda H}{2\lambda}t\right)\left\{-\frac{(\lambda e_{pq}+g_{pq})}{\lambda}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right.$$
$$+\frac{2\lambda(\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(f_{pq}-\omega g_{pq})}{\lambda\omega b_{pq}}$$
$$\left.\left.+\frac{\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right\}\right), \quad (60)$$

$$\tau_{2c}(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\left(e_{pq}\cos(\omega t)-f_{pq}\sin(\omega t)+\frac{g_{pq}}{\lambda}\exp(-\frac{t}{\lambda})\right.$$
$$+\exp\left(-\frac{1+\lambda H}{2\lambda}t\right)\left\{-\frac{(\lambda e_{pq}+g_{pq})}{\lambda}\cosh\left(\frac{b_{pq}}{2\lambda}t\right)\right.$$
$$+\{\frac{2\lambda(\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(f_{pq}-\omega g_{pq})}{\lambda\omega b_{pq}}$$
$$\left.\left.+\frac{\omega(1+\lambda H)(\lambda e_{pq}+g_{pq})}{\lambda\omega b_{pq}}\}\sinh\left(\frac{b_{pq}}{2\lambda}t\right)\right\}\right), \quad (61)$$

for cosine and

$$\tau_{1s}(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big(e_{pq}\sin(\omega t)+f_{pq}\cos(\omega t)+\frac{h_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{-\frac{\lambda f_{pq}+h_{pq}}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\{\frac{H+\frac{\nu\phi}{k}-2\lambda(\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(e_{pq}+\omega h_{pq})}{\lambda\omega b_{pq}}$$
$$+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big),\qquad(62)$$

$$\tau_{2s}(x,y,t)=-\frac{16\mu U}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\Big(e_{pq}\sin(\omega t)+f_{pq}\cos(\omega t)+\frac{h_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{-\frac{\lambda f_{pq}+h_{pq}}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\{\frac{H+\frac{\nu\phi}{k}-2\lambda(\nu\lambda_{pq}+H+\frac{\nu\phi}{k})(e_{pq}+\omega h_{pq})}{\lambda\omega b_{pq}}$$
$$+\frac{\omega(1+\lambda H)(\lambda f_{pq}+h_{pq})}{\lambda\omega b_{pq}}\}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big).\qquad(63)$$

for sine oscillations of rectangular duct for Maxwell fluid.

6. Calculation of the volume flow rate

The volume flow rate for cosine oscillations of the rectangular duct is given by

$$Q_c(t)=\int_0^d\int_0^h u(x,y,t)dxdy\,.$$

Inserting $u(x,x,t)$ from Eq. (31) into the above relation, we find the volume flow rate

$$Q_c(t)=Uhd\cos(\omega t)-\frac{64U}{dh}\sum_{m,n=0}^{\infty}\frac{1}{(\lambda_p\mu_q)^2}$$
$$\times\Big\{c_{pq}\cos(\omega t)-d_{pq}\sin(\omega t)+\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\{(1-c_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(1-c_{pq})(1+\lambda H)+2\lambda\omega d_{pq}}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\}\Big\},\qquad(64)$$

for cosine oscillations of the rectangular duct for Maxwell fluid.

Similarly, we obtain the volume flow rate

$$Q_s(t)=Uhd\sin(\omega t)-\frac{64U}{dh}\sum_{m,n=0}^{\infty}\frac{1}{(\lambda_p\mu_q)^2}$$
$$\times\Big\{c_{pq}\sin(\omega t)+d_{pq}\cos(\omega t)-\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\{d_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{d_{pq}(1+\lambda H)-2\lambda\omega(1-c_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\}\Big\}.\qquad(65)$$

for the sine oscillations of the rectangular duct for Maxwell fluid.

7. Flow through a rectangular duct oscillating due to pressure gradient

Consider a Maxwell fluid at rest in a duct of rectangular cross-section whose sides are at $x=0, x=d$, $y=0$ and $y=h$. At time $t=0^+$ an oscillating pressure gradient is applied to the fluid in the z-direction. The governing equations, initial and boundary conditions for this problem corresponding to Eqs. (13)- (15) become

$$(1+\lambda\frac{\partial}{\partial t})\frac{\partial F(x,y,t)}{\partial t}=-\frac{1}{\rho}(1+\lambda\frac{\partial}{\partial t})\frac{\partial P}{\partial z}$$
$$+\nu\Big(\frac{\partial^2F(x,y,t)}{\partial x^2}+\frac{\partial^2F(x,y,t)}{\partial y^2}\Big)$$
$$-H(1+\lambda\frac{\partial}{\partial t})F(x,y,t)-\frac{\nu\phi}{k}F(x,y,t),\qquad(66)$$

$$F(x,y,0)=\frac{\partial F(x,y,0)}{\partial t}=0$$
$$\text{for }(x,y)\in(0,d)\times(0,h),\qquad(67)$$
$$F(0,y,t)=F(d,y,t)=F(x,0,t)=F(x,h,t)=0\ \forall\ t.\qquad(68)$$

Let us assume that at time $t=0^+$, a pressure gradient is of the form

$$\frac{\partial P}{\partial z}=-\rho Q\exp(i\omega t),\qquad(69)$$

where Q is amplitude and ω is the frequency of oscillation.

Using Eq. (69) in Eq. (66), we obtain

$$(1+\lambda\frac{\partial}{\partial t})\frac{\partial F(x,y,t)}{\partial t}=Q(1+\lambda\frac{\partial}{\partial t})\exp(i\omega t)$$
$$+\nu\Big(\frac{\partial^2F(x,y,t)}{\partial x^2}+\frac{\partial^2F(x,y,t)}{\partial y^2}\Big)$$
$$-H(1+\lambda\frac{\partial}{\partial t})F(x,y,t)-\frac{\nu\phi}{k}F(x,y,t).\qquad(70)$$

Employing the same methodology as in the previous case, we find the corresponding expressions of the velocity field under the form

$$u(x,y,t)=Q\cos(\omega t)-\frac{16Q}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big\{\alpha_{pq}\cos(\omega t)-\beta_{pq}\sin(\omega t)+\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\{(1-\alpha_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(1-\alpha_{pq})(1+\lambda H)+2\lambda\omega\beta_{pq}}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big\}, \quad (71)$$

and

$$v(x,y,t)=Q\sin(\omega t)-\frac{16Q}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big\{\alpha_{pq}\sin(\omega t)+\beta_{pq}\cos(\omega t)-\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\Big(\beta_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{\beta_{pq}(1+\lambda H)-2\lambda\omega(1-\alpha_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\Big\}. \quad (72)$$

where

$$D_{mn}=\alpha_{mn}+i\beta_{mn}, \quad (73)$$

α_{mn} and β_{mn} are given by

$$\alpha_{mn}=1-\frac{\lambda^2\omega^2H+Z_{mn}}{\left(-\lambda\omega^2+Z_{mn}\right)^2+\omega^2(1+\lambda H)^2}, \quad (74)$$

$$\beta_{mn}=\frac{\omega\Big(1+\lambda^2\omega^2-\lambda\nu(\lambda_{mn}+\frac{\phi}{k})\Big)}{\left(-\lambda\omega^2+K_{mn}\right)^2+\omega^2(1+\lambda H)^2}. \quad (75)$$

By taking $t\to 0$ in Eqs. (71) and (72), we obtain the following transient components

$$u_t(x,y,t)=-\frac{16Q}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big\{\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\Big((1-\alpha_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(1-\alpha_{pq})(1+\lambda H)+2\lambda\omega\beta_{pq}}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\Big\}, \quad (76)$$

$$v_t(x,y,t)=-\frac{16Q}{dh}\sum_{m,n=0}^{\infty}\frac{\sin(\lambda_p x)}{\lambda_p}\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big\{\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)\Big(-\beta_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$-\frac{\beta_{pq}(1+\lambda H)-2\lambda\omega(1-\alpha_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\Big\}, \quad (77)$$

for $1+\lambda H>0$.

Adopting a similar procedure as before, the expressions for the tangential tensions are given by

$$\tau_1(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big((\gamma_{pq}+i\delta_{pq})\exp(i\omega t)+\frac{\xi_{pq}+i\psi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)$$
$$\times\Big\{-\frac{(\lambda\gamma_{pq}+\xi_{pq})+i(\lambda\delta_{pq}+\psi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\Big(\frac{(\lambda\gamma_{pq}+\xi_{pq})(1-4\lambda+\lambda H(1-2\lambda))}{\lambda b_{pq}}$$
$$+\frac{2\lambda^2(1-2\omega\psi_{pq})}{\lambda b_{pq}}\Big)\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+i\frac{(\lambda\delta_{pq}+\psi_{pq})\Big(1-4\lambda+\lambda H(1-2\lambda)\Big)+2\lambda^2\omega}{\lambda b_{pq}}$$
$$\times\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big), \quad (78)$$

$$\tau_2(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\Big((\gamma_{pq}+i\delta_{pq})\exp(i\omega t)+\frac{\xi_{pq}+i\psi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)$$
$$\times\Big\{-\frac{(\lambda\gamma_{pq}+\xi_{pq})+i(\lambda\delta_{pq}+\psi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\Big(\frac{(\lambda\gamma_{pq}+\xi_{pq})(1-4\lambda+\lambda H(1-2\lambda))}{\lambda b_{pq}}$$
$$+\frac{2\lambda^2(1-2\omega\psi_{pq})}{\lambda b_{pq}}\Big)\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+i\frac{(\lambda\delta_{pq}+\psi_{pq})\Big(1-4\lambda+\lambda H(1-2\lambda)\Big)+2\lambda^2\omega}{\lambda b_{pq}}$$
$$\times\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big), \quad (79)$$

where

$$E_{pq}=\gamma_{pq}+i\delta_{pq}, \qquad F_{pq}=\xi_{pq}+i\psi_{pq}, \quad (80)$$

γ_{pq}, δ_{pq}, ξ_{pq} and ψ_{pq} are given by

$$\gamma_{pq}=\frac{\lambda\omega^2\left(1+\lambda\omega^2-2\nu(\lambda_{pq}+\frac{\phi}{k})\right)+\omega^2}{[-\lambda\omega^2(2+\lambda H)+Z_{pq}]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}$$
$$+\frac{\omega^2\left(\lambda^2(H^2+\lambda\omega^2)\right)+Z_{pq}\left(Z_{pq}-1-\lambda^2\omega^2\right)}{[-\lambda\omega^2(2+\lambda H)+Z_{pq}]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}, \quad (81)$$

$$\delta_{pq}=\frac{\omega(\lambda^2\omega^2\left(1+2\nu\lambda_{pq}+\frac{2\nu\phi}{k}+\lambda(1-H-\omega^2)\right))}{[-\lambda\omega^2(2+\lambda H)+Z_{pq}]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}$$
$$+\frac{\omega(1-\lambda Z_{pq}{}^2)}{[-\lambda\omega^2(2+\lambda H)+Z_{pq}]^2+\omega^2[1+\lambda(H+Z_{pq}-\lambda\omega^2)]^2}, \quad (82)$$

$$\xi_{pq}=\frac{\lambda(1-\nu\lambda_{pq}+\lambda^2\omega^2-\frac{\nu\phi}{k})}{\nu(\frac{\phi}{k}+\lambda_{pq})(1+\lambda^2\omega^2)},\ \psi_{pq}=\frac{\omega\lambda^2}{1+\lambda^2\omega^2}. \quad (83)$$

From Eqs. (78) and (79), we obtain the tangential tensions

$$\tau_{1c}(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big(\gamma_{pq}\cos(\omega t)-\delta_{pq}\sin(\omega t)+\frac{\xi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{\frac{(\lambda\gamma_{pq}+\xi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\Big(\frac{(\lambda\gamma_{pq}+\xi_{pq})(1-4\lambda+\lambda H(1-2\lambda))}{\lambda b_{pq}}$$
$$+\frac{2\lambda^2(1-2\omega\psi_{pq})}{\lambda b_{pq}}\Big)\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big), \quad (84)$$

$$\tau_{2c}(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\Big(\gamma_{pq}\cos(\omega t)-\delta_{pq}\sin(\omega t)+\frac{\xi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{\frac{(\lambda\gamma_{pq}+\xi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\Big(\frac{(\lambda\gamma_{pq}+\xi_{pq})(1-4\lambda+\lambda H(1-2\lambda))}{\lambda b_{pq}}$$
$$+\frac{2\lambda^2(1-2\omega\psi_{pq})}{\lambda b_{pq}}\Big)\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big), \quad (85)$$

corresponding to cosine and

$$\tau_{1s}(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\lambda_p x)\frac{\sin(\mu_q y)}{\mu_q}$$
$$\times\Big(\gamma_{pq}\sin(\omega t)+\delta_{pq}\cos(\omega t)+\frac{\psi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{-\frac{(\lambda\delta_{pq}+\psi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(\lambda\delta_{pq}+\psi_{pq})\left(1-4\lambda+\lambda H(1-2\lambda)\right)+2\lambda^2\omega}{\lambda b_{pq}}$$
$$\times\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big), \quad (86)$$

$$\tau_{2s}(x,y,t)=-\frac{16\mu Q}{dh}\sum_{m,n=0}^{\infty}\cos(\mu_q y)\frac{\sin(\lambda_p x)}{\lambda_p}$$
$$\times\Big(\gamma_{pq}\sin(\omega t)+\delta_{pq}\cos(\omega t)+\frac{\psi_{pq}}{\lambda}\exp(-\frac{t}{\lambda})$$
$$+\exp\Big(-\frac{1+\lambda H}{2\lambda}t\Big)\Big\{-\frac{(\lambda\delta_{pq}+\psi_{pq})}{\lambda}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(\lambda\delta_{pq}+\psi_{pq})\left(1-4\lambda+\lambda H(1-2\lambda)\right)+2\lambda^2\omega}{\lambda b_{pq}}$$
$$\times\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big\}\Big). \quad (87)$$

sine oscillations due to oscillating pressure gradient.

The volume flow rates for cosine and sine oscillations induced by oscillating pressure gradient are given by

$$Q_c(t)=Qhd\cos(\omega t)-\frac{64Q}{dh}\sum_{m,n=0}^{\infty}\frac{1}{(\lambda_p\mu_q)^2}$$
$$\times\Big\{\alpha_{pq}\cos(\omega t)-\beta_{pq}\sin(\omega t)+\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\Big((1-\alpha_{pq})\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{(1-\alpha_{pq})(1+\lambda H)+2\lambda\omega\beta_{pq}}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\Big\}, \quad (88)$$

$$Q_s(t)=Qhd\sin(\omega t)-\frac{64Q}{dh}\sum_{m,n=0}^{\infty}\frac{1}{(\lambda_p\mu_q)^2}$$
$$\times\Big\{\alpha_{pq}\sin(\omega t)+\beta_{pq}\cos(\omega t)-\exp\Big(-\frac{(1+\lambda H)t}{2\lambda}\Big)$$
$$\times\Big(\beta_{pq}\cosh\Big(\frac{b_{pq}}{2\lambda}t\Big)$$
$$+\frac{\beta_{pq}(1+\lambda H)-2\lambda\omega(1-\alpha_{pq})}{b_{pq}}\sinh\Big(\frac{b_{pq}}{2\lambda}t\Big)\Big)\Big\}. \quad (89)$$

8- Results and discussion

This section displays the graphical illustration of the velocity fields, the corresponding tangential tensions and volume flow rate for the flow discussed above. Figs. 1-3 are sketched to notice the effect of material parameters on the velocity profile for sine oscillations of Maxwell induced by oscillating pressure gradient. In Fig. 1, transient velocity profile is plotted against time for various values of relaxation time λ. It is seen that the amplitude of oscillation of velocity increases and the required time to reach the steady state also increases as λ increases. The decay of transient velocity in time is plotted in Fig. 2 for different values of angular frequency ω. From this Fig., it is concluded that the amplitude of oscillation as well as the time to reach the steady increases for increasing ω. From Fig. 3, it is seen that the amplitude of oscillation increases with increase in ν.

Now we intent to investigate the magnetic and porosity effects on the velocity, tangential tension and volume flow rate for the two flow problems. In Figs. 4-9 panel (a) shows the effect of magnetic parameter and panel (b) the effect of porosity of medium. In Fig. 4, we plotted transient velocity profiles for sine oscillations of rectangular duct against time. It predicts that the transient velocity decreases with increase in the magnetic parameter and porosity of medium and the required time to reach the steady state also decreases. Fig. 5 shows that magnetic parameter and porosity of medium decrease the transient velocity profiles for the case of sine oscillations induced by oscillating pressure gradient.

From Figs. 6 and 7, it is seen that the volume flow rate decreases with the increase of magnetic parameter and porosity of medium for the both flow problems. Figs. 8 and 9 are sketched to demonstrate the velocity changes with the magnetic parameter and porosity of medium for sine oscillations of the duct and sine oscillations induced by oscillating pressure gradient. It is seen that amplitude of oscillation of velocity in magnitude decreases with the increase of these parameters for the both problems but periodicity remains the same.

The fluctuations of the tangential tensions for sine oscillations of rectangular duct, τ_{1s}, verses time are shown in Figs. 10 and 11 for H and ϕ respectively. From Fig. 10(a) it is noted that the tangential tension, τ_{1s}, changes its monotony with respect to H for small time. For large time, when the flow reaches the steady state amplitude of oscillation of tangential tension, τ_{1s}, increases with increasing H as depicted in Fig. 10(b). Similarly, Fig. 11 shows that the effect of ϕ on the tangential tension, τ_{1s}, is the same as that of H. The effect of H and ϕ on the tangential tension, τ_{1s}, for the case of sine oscillations induced by oscillating pressure gradient is shown in Fig. 12 and it is seen that there is no significant effect initially and later for any time tangential tension, τ_{1s} increases with the increase of these parameters.

9- CONCLUSIONS

Unsteady flow of Maxwell fluid through porous rectangular duct in the presence of magnetic field has been studied for two flow situations by means of the double finite Fourier sine and Laplace transforms. The relaxation time, frequency of oscillation and kinematic viscosity increase the amplitude of oscillation of velocity. The transient velocity profile and the time to reach the steady state decrease with the increase of magnetic field strength and porosity of medium. Volume flow rate decreases while the tangential tension increases with the increase of H and ϕ. In the special cases, we can obtain the solutions corresponding to the Maxwell and Newtonian fluids. We can obtain the corresponding solutions for Maxwell fluid by substituting $H=0$ and $\phi=0$, for Maxwell fluid in the presence of magnetic field by substituting $\phi=0$ and for Newtonian fluid by substituting $H=0$, $\phi=0$ and $\lambda\to 0$.

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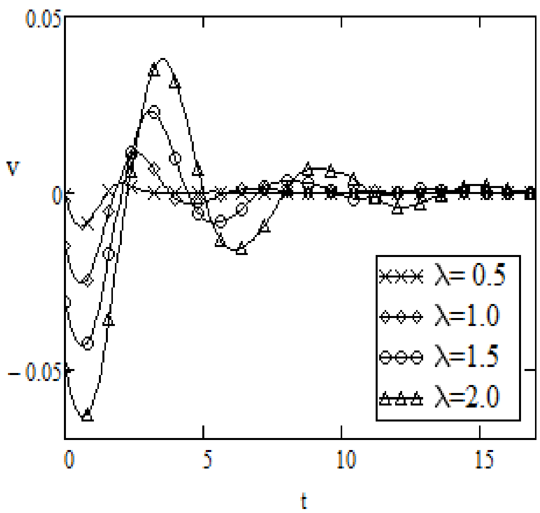

Fig. 1 Transient velocity profiles of sine oscillations given in Eq. (36) for Maxwell fluid induced by oscillating pressure gradient for different values of λ. Other parameters and values are taken as $U=0.1$, $d=1$, $h=2$, $x=0.5$, $y=1$, $k=0.01$, $\phi=0$, $M=0$, $\omega=0.5$ and $\nu=0.2$

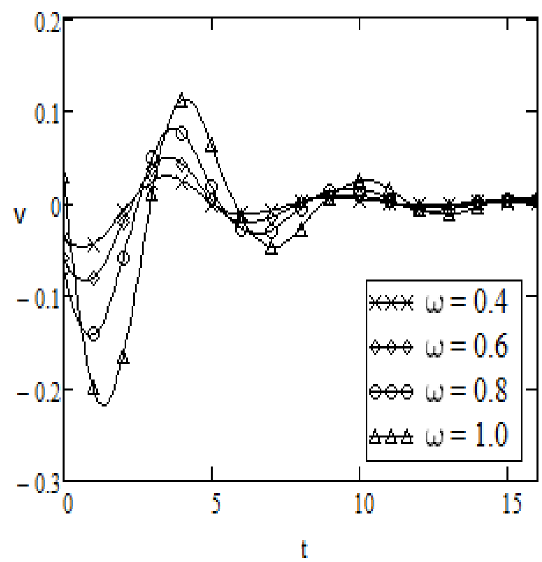

Fig. 2 Transient velocity profiles of sine oscillations given in Eq. (36) for Maxwell fluid induced by oscillating pressure gradient for different values of ω. Other parameters and values are taken as $U=0.1$, $d=1$, $h=2$, $x=0.5$, $y=1$, $k=0.01$, $M=0$, $\phi=0$, $\lambda=2$ and $\nu=0.2$.

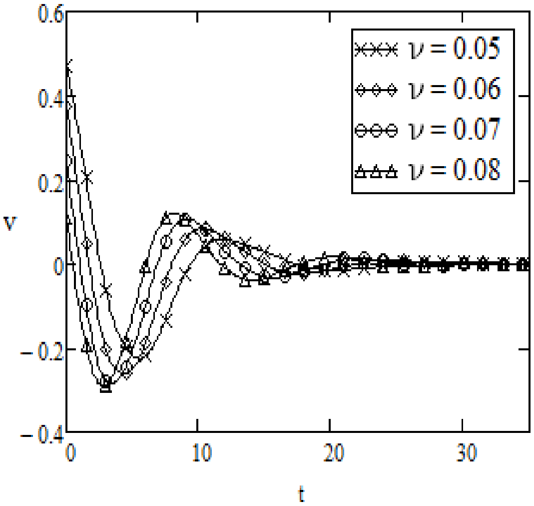

Fig. 3 Transient velocity profiles of sine oscillations given in Eq. (36) for Maxwell fluid induced by oscillating pressure gradient for different values of ν. Other parameters and values are taken as $U=0.1$, $d=1$, $h=2$, $x=0.5$, $y=1$, $k=0.01$, $M=0$, $\phi=0$, $\lambda=3$ and $\omega=0.5$

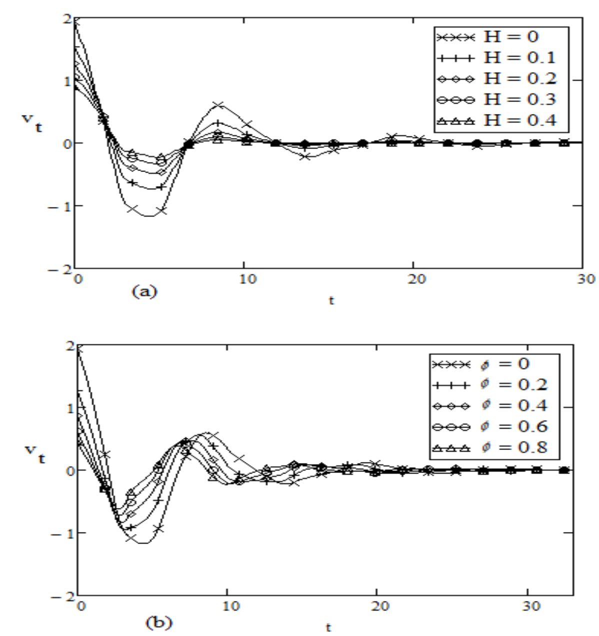

Fig. 4 Transient velocity profiles of sine oscillations given in Eq. (36) for Maxwell fluid in oscillating rectangular duct for different values of (a) magnetic parameter and (b) porosity of medium. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.5$, $y=1$, $k=0.01$, $\lambda=3$, $\omega=0.5$ and $\nu=0.1$.

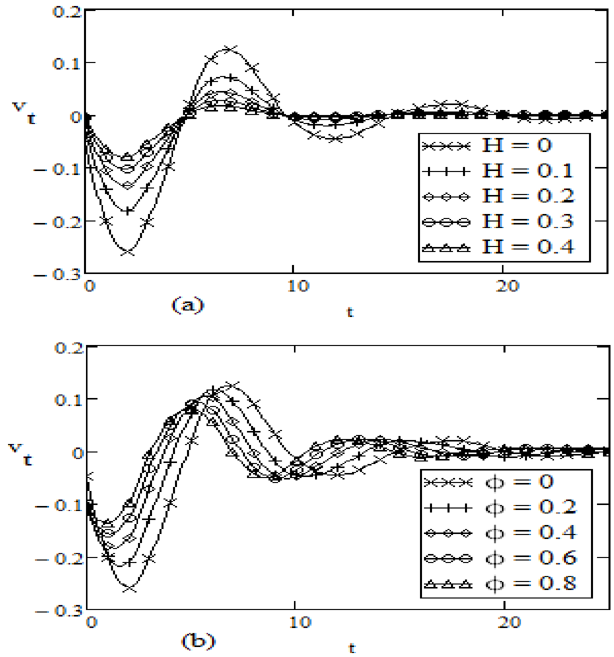

Fig. 5 Transient velocity profiles of sine oscillations induced by oscillating pressure gradient given in Eq. (77) for different values of (a) magnetic parameter and (b) porosity of medium. Other parameters and values are taken as $U=0.1$, $d=1$, $h=2$, $x=0.5$, $y=1$, $k=0.1$, $\omega=0.5$ and $\nu=0.1$.

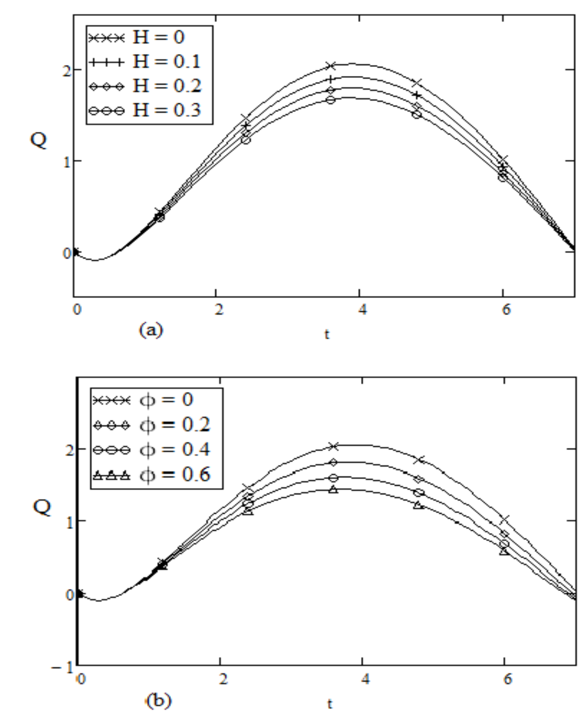

Fig. 6 Volume flow rate of sine oscillations given in Eq. (65) for Maxwell fluid in oscillating rectangular duct for different values of (a) magnetic parameter and (b) porosity parameter. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.1$, $y=0.2$, $k=0.1$, $\lambda=0.5$, $\omega=0.5$ and $\nu=0.1$.

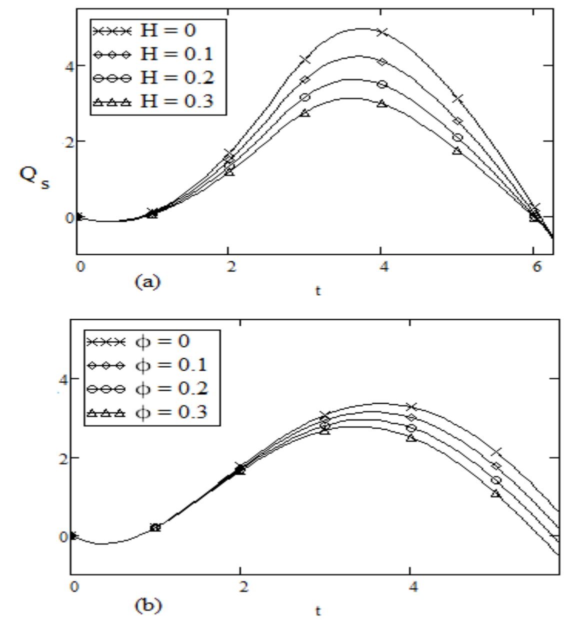

Fig. 7 Volume flow rate of sine oscillations induced by oscillating pressure gradient given in Eq. (89) for different values of (a) magnetic parameter and (b) porosity of medium. Other parameters and values are taken as $U=2, d=1$, $h=2$, $x=0.5$, $y=0.5$, $k=0.1$, $\lambda=3$ $\omega=0.5$ and $\nu=0.1$.

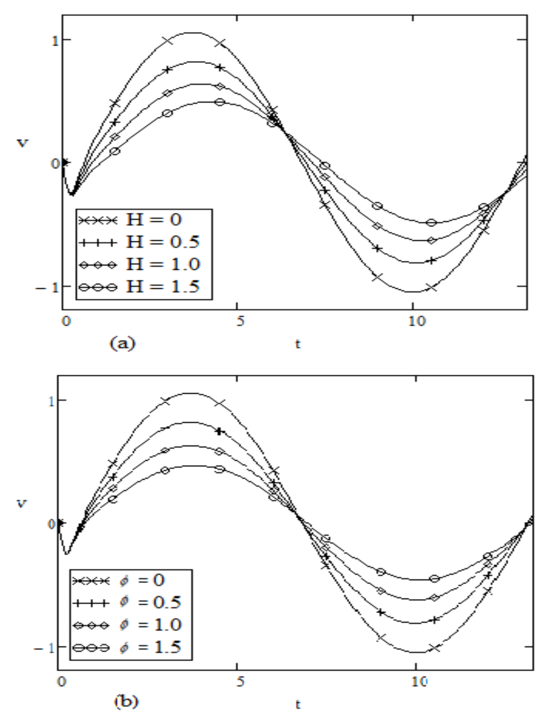

Fig. 8 Velocity profiles of sine oscillations of oscillating rectangular duct given in Eq. (32) for Maxwell fluid for different values of (a) magnetic parameter and (b) porosity of medium. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.1$, $y=0.2$, $k=0.1, \lambda=0.5$, $\omega=0.5$ and $\nu=0.1$.

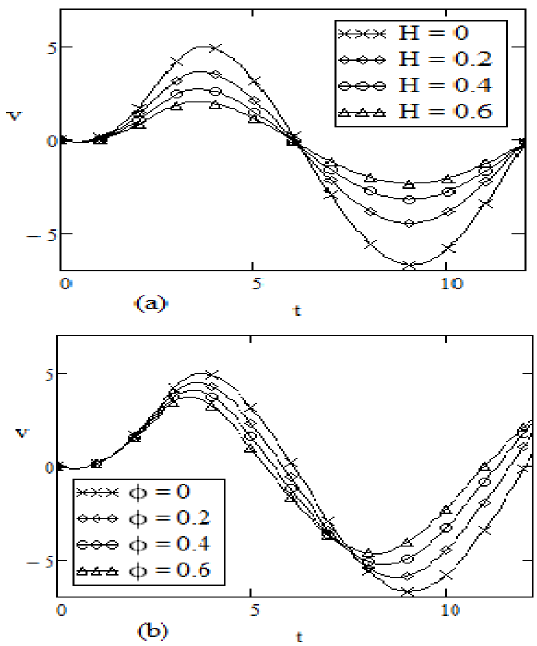

Fig. 9 Velocity profiles of sine oscillations induced by oscillating pressure gradient given in Eq. (72) for different values of (a) magnetic parameter and (b) porosity of medium. Other parameters and values are taken as $U=2$, $d=1$, $h=2$, $x=0.5$, $y=0.5$, $k=0.1$, $\lambda=3$, $\omega=0.5$ and $\nu=0.1$.

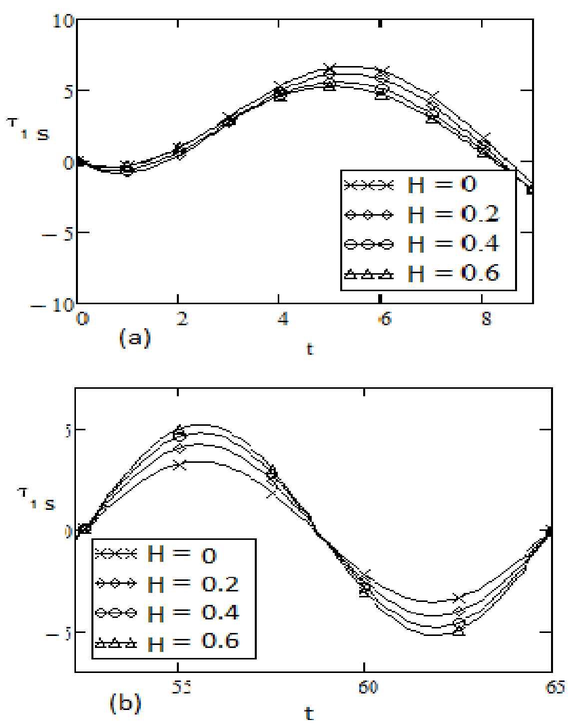

Fig. 10 Tangential tensions τ_{1s} of sine oscillations of rectangular duct (Eq. (62)) for Maxwell fluid for different values of H. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.5$, $y=0.5$, $k=0.1$, $\lambda=3$, $\omega=0.5$ and $\nu=0.0012$.

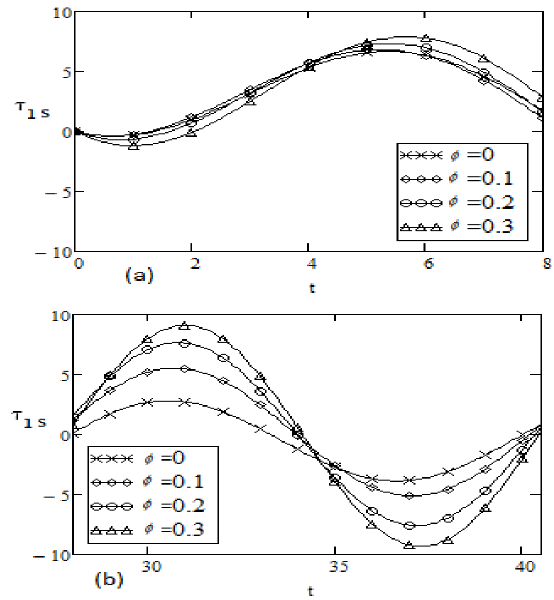

Fig. 11 Tangential tensions τ_{1s} of sine oscillations of rectangular duct (Eq. (62)) for Maxwell fluid for different values of ϕ. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.5$, $y=0.5$, $k=0.1$, $\lambda=3$, $\omega=0.5$ and $\nu=0.0012$.

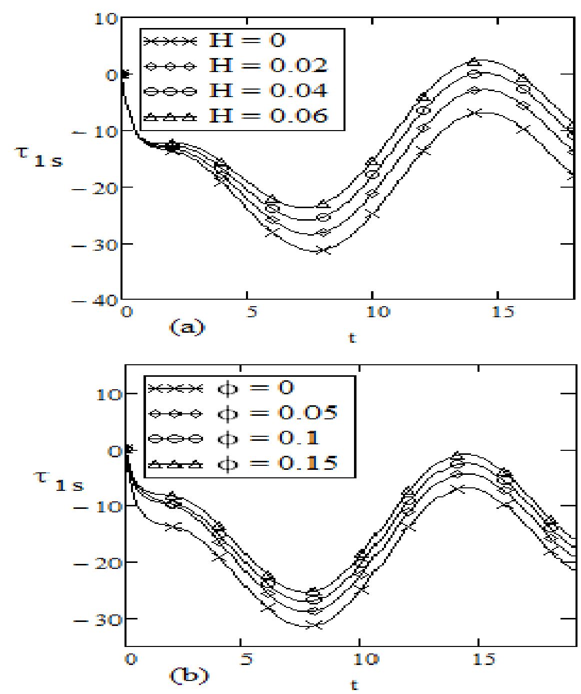

Fig. 12 Tangential tensions τ_{1s} of sine oscillations induced by oscillating pressure gradient (Eq. (86)) for different values of H and ϕ. Other parameters and values are taken as $U=1$, $d=1$, $h=2$, $x=0.5$, $y=0.5$, $k=0.01$, $\lambda=0.5$, $\omega=0.5$ and $\nu=0.0012$.

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